PMM U.S.S.R., Vol.49, No.3, pp. 387-389, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON THE NON-EXISTENCE OF THE ADDITIONAL INTEGRAL IN THE PROBLEM OF THE MOTION OF A HEAVY RIGID ELLIPSOID ALONG A SMOOTH PLANE*

A.A. BUROV and A.V. KARAPETYAN

Two cases are known, in which the system of equations of motion of a heavy rigid body along a smooth plane is Liouville integrable (see e.g. /1/); in one the body is a sphere whose centre of mass coincides with its geometrical centre and the moments of inertia are arbitrary, and in the other the body is a solid of revolution. In the present paper the necessary conditions for an additional first integral to exist, analytic with respect to the plane variables, is obtained for ellipsoidal bodies resembling a sphere whose principal moments of inertia are different and whose centre of mass coincides with the geometrical centre.

1. Consider the problem of the motion of a heavy rigid body along a fixed, perfectly smooth horizontal plane. We shall specify the position of the body by means of the coordinates x, y of its centre of mass G in a fixed $Oxyz$ coordinate system (the Oxy plane coincides with the supporting plane and the Oz axis is directed vertically upwards), and the Euler angles φ, ψ, θ, which determine the orientation of the principal central axes of inertia of the body $G\xi_1, G\xi_2, G\xi_3$ relative to the axes of the fixed coordinate system.

The body on a smooth plane represents a conservative holonomic system, and its motion is determined from the Lagrange equations with the Lagrangian /2/

$$\begin{aligned}L = {}^1/_2\{(I_3 + m\chi_2^2 \sin^2\theta)\varphi^{\cdot 2} + [(I_1 \sin^2\varphi + I_2\cos^2\varphi)\sin^2\theta + \\ I_3\cos^2\theta]\psi^{\cdot 2} + [I_1\cos^2\varphi + I_2\sin^2\varphi + m(\chi_1\cos\theta - \xi_3\sin\theta)]\theta^{\cdot 2} + \\ 2I_3\cos\theta\varphi^{\cdot}\psi^{\cdot} + 2m(\chi_1\cos\theta - \xi_3\sin\theta)\chi_2\sin\theta\varphi^{\cdot}\theta^{\cdot} + \\ 2(I_1 - I_2)\sin\theta\sin\varphi\cos\varphi\psi^{\cdot}\theta^{\cdot}\} + \frac{m}{2}(x^{\cdot 2} + y^{\cdot 2}) + mg(\chi_1\sin\theta + \xi_3\cos\theta)\end{aligned}$$

$$\chi_1 = \xi_1\sin\varphi + \xi_2\cos\varphi,\quad \chi_2 = \xi_1\cos\varphi - \xi_2\sin\varphi$$

Here I_1, I_2, I_3 are the principal central moments of inertia of the body, m is its mass, g is the acceleration due to gravity, and ξ_1, ξ_2, ξ_3 are the coordinates of the point of contact between the body and the plane in the principal central axes of inertia, representing the functions of the variables θ and φ, determined from the equations of the body surface.

Changing from the generalized coordinates and velocities to the coordinates and moments, we can write the equations of motion of the body on a smooth plane in canonical form, with the Hamiltonian function

$$H = \frac{1}{2} < p,\ A^{-1}p > + \frac{1}{2m}(p_x^2 + p_y^2) - mg(\chi_1\sin\theta + \xi_3\cos\theta)$$

where $p = (p_\varphi, p_\psi, p_\theta)$, A is the matrix of the quadratic form in $\varphi^{\cdot}, \psi^{\cdot}, \theta^{\cdot}$, appearing within the curly brackets in the expression for the function L.

Since the function H is independent of x, y, ψ, it follows that the system in question has, in addition to the energy integral $H = \text{const}$, another three integrals

$$p_x = P_x,\ p_y = P_y,\ p_\psi = P_\psi$$

and we can assume without loss of generality that the constants P_x and P_y are zero, i.e. the projection of the centre of mass of the body on the supporting surface is immobile (the constant P_ψ is arbitrary).

In the general case, the full integrability of the equations of motion of the body on a smooth plane requires, as in the case of the motion of a heavy rigid body with a fixed point, one additional integral. We note that unlike the latter case, the form of the body surface plays a significant role in the problem in question.

2. Let the body be bounded by an ellipsoidal surface. Then the coordinates of the point of contact between the body and the plane can be written in the form (from now on we shall assume that the summation over the repeated indices is carried out from 1 to 3)

$$\xi_k = -\alpha_k - \rho^{-1}\rho_i^2\Gamma_i c_{ik}\quad (k = 1, 2, 3)\qquad \rho^2 = \rho_i^2\Gamma_i^2,\quad \Gamma_i = c_{ij}\gamma_j$$

*Prikl.Matem.Mekhan., 49, 3, 501-503, 1985

Here ρ_i denote the semi-axes of the ellipsoid forming the surface of the body, c_{ij} are the cosines of the angles between its principal axes and the principal axes of inertia of the body, $G\xi_j, -\alpha_j$ and γ_j are the corresponding coordinates of the geometrical centre of the ellipsoid in the principal central axes of inertia of the body and of the projection of the unit vector of the vertial on these axes

$$\gamma_1 = \sin\theta\sin\varphi,\ \gamma_2 = \sin\theta\cos\varphi,\ \gamma_3 = \cos\theta$$

The potential energy of the body will now take the form

$$U = mg\,[\alpha_j\gamma_j + (\rho_i^2\Gamma_i^2)^{1/2}]$$

Let us assume that the ellipsoidal body in question is nearly sperhical, with the centre of mass of this sphere coinciding with its geometrical centre, and that $I_1 < I_2 < I_3$. Then

$$\alpha_k = \varepsilon a_k,\ \rho_k = r + \varepsilon r_k \tag{2.1}$$

When $\varepsilon = 0$ the problem is integrable (see the first case of integrability). Let us expand the Hamiltonian function in powers of the parameter ε

$$H = H_0 + \varepsilon H_1 + \ldots \tag{2.2}$$
$$H_1 = mg\,[a_j\gamma_j + r_i\Gamma_i^2]$$

Here H_0 is identical, apart from an additive constant, with the kinetic energy of motion of a rigid body about a fixed point.

3. If all $r_i = 0$, and $a_1^2 + a_2^2 + a_3^2 \neq 0$ (the body degenerates into a sphere whose centre of mass does not coincide with its geometrical centre), then H_1 has a form analogous to that of H_1 in the case of perturbation of the classical Euler-Poinsot equation and, according to /3/, when $\varepsilon \neq 0$ are small, the equations of motion of the body in question have no additional first integral analytic with respect to the phase variables.

4. Suppose now that all $a_j = 0$, $r_1^2 + r_2^2 + r_3^2 \neq 0$ (the body is an ellipsoid whose centre of mass coincides with its geometrical centre). Then H_1 will be a quadratic form in γ_i and the problem will reduce to that of the existence of a complementary integral in the problem of the motion of a rigid body about a fixed point in a field with a quadratic potential.

If the moments of inertia of the body are all different, then, according to /4/ the necessary conditions for the complementary integral to exist are the same in this case as the Klebsch conditions. For the problem in question these conditions take the form

$$r_ic_{i1}c_{i2} = 0,\ r_ic_{i2}c_{i3} = 0,\ r_ic_{i3}c_{i1} = 0 \tag{4.1}$$
$$I_1r_i\,(c_{i2}^2 - c_{i3}^2) + I_2r_i\,(c_{i3}^2 - c_{i1}^2) + I_3r_i\,(c_{i1}^2 - c_{i2}^2) = 0 \tag{4.2}$$

Relations (4.1) represent a linear homogeneous system in r_1, r_2, r_3. It can be shown that the determinant of the matrix

$$C = \begin{Vmatrix} c_{11}c_{12} & c_{21}c_{22} & c_{31}c_{32} \\ c_{12}c_{13} & c_{22}c_{23} & c_{32}c_{33} \\ c_{13}c_{11} & c_{23}c_{21} & c_{33}c_{31} \end{Vmatrix}$$

is zero, and consequently system (4.1) has non-trivial solutions.

We shall show that, when condition (4.2) holds, $\operatorname{rank} C = 0$.

Let $\operatorname{rank} C = 2$, i.e. the inertia ellipsoid and the surface ellipsoid have no common axes. The general solution of system (4.1) has the form

$$r_1 = r_2 = r_3 = p \tag{4.3}$$

It follows that the surface ellipsoid is a sphere, which contradicts the assumption that the inertia and the surface ellipsoids have no common axes (solution (4.3) transforms the equation (4.2) into an identity).

Let us assume that $\operatorname{rank} C = 1$. It can be shown that in this case the matrix of the cosines of the angles between the principal axes of the inertia ellipsoid and the body surface ellipsoid have, apart from the numering and the choice of the positive directions of the axes, the form

$$c_{11} = 1,\ c_{12} = c_{13} = c_{21} = c_{31} = 0$$
$$c_{22} = c_{33} = \cos\delta,\ c_{23} = -c_{32} = -\sin\delta\ (0 < \delta < \pi/2)$$

Here the inertia and the surface ellipsoid have one common axis, and the other two axes of the surface ellipsoid are rotated by an angle δ relative to the corresponding axes of the inertia ellipsoid. Such a mass distribution is characteristic of the celtic stone. In this case the general solution of system (4.1) has the form

$$r_1 = s,\ r_2 = r_3 = p$$

and condition (4.2) is now written in the form

$$(I_3 - I_2)(s - p) = 0 \tag{4.4}$$

Since $I_3 \neq I_2$, it follows from (4.4) that $s = p$ and the surface ellipsoid, as in the previous case, is a sphere, which contradicts the condition $0 < \delta < \pi/2$. Therefore $\operatorname{rank} C = 0$. Here the axes of the inertia ellipsoid and the surface ellipsoid coincide, conditions (4.1) are satisfied identically, and relation (4.2) takes the form

$$I_1(r_2 - r_3) + I_2(r_3 - r_1) + I_3(r_1 - r_2) = 0 \qquad (4.5)$$

The above relation certainly holds in the first, as well as in the second case of integrability, though the conditions for a complementary integral to exist in these cases ($r_1 = r_2 = r_3$ or $I_1 = I_2$, $r_1 = r_2$) are not necessary for the relation to hold. We also note that relation (4.5) does not hold for a homogeneous ellipsoidal body.

5. In the case of an arbitrarily small perturbation in the integrable problem of the motion of a sphere whose centre of mass coincides with its geometrical centre, the necessary conditions for a complementary first integral to exist, analytic in the phase variables, in the class of bodies with ellipsoidal surfaces (a_k and r_k in (2.1) are such, that $a_1^2 + a_2^2 + a_3^2 \neq 0$, $r_1^2 + r_2^2 + r_3^2 \neq 0$), are combinations of the corresponding conditions of Sect.3 and 4. This proves the following

Theorem. The following three conditions are simultaneously necessary for a complementary first integral to exist, analytic in the phase variables, in the problem of the motion of a heavy rigid ellipsoidal, nearly spherical body, whose centre of mass coincides with its geometrical centre and the moments of inertia are all different: 1) the centre of mass of the ellipsoid coincides with its geometrical centre; 2) the principal axes of the inertia ellipsoid and surface ellipsoid coincide; 3) the moments of inertia of the ellipsoid and the semi-axes of its surface are connected by the relation

$$I_1(\rho_2 - \rho_3) + I_2(\rho_3 - \rho_1) + I_3(\rho_1 - \rho_2) = 0$$

The problem of the existence of a complementary analytic integral in the problem of the motion of a body of arbitrary, nearly spherical shape, whose centre of mass coincides with its geometrical centre, is more interesting and more complex. In this case, the first approximation in terms of a small parameter already yields a potential which may represent, generally speaking, an arbitrary function of the direction cosines $\gamma_1, \gamma_2, \gamma_3$, unlike the function H_1 (2.2) representing the sum of the linear and quadratic forms of the variables $\gamma_1, \gamma_2, \gamma_3$.

REFERENCES

1. APPEL' P. Theoretical Mechanics. Vol.2, Moscow, Fizmatgiz, 1960.
2. KARAPETYAN A.V., On the stability of the steady motions of a heavy rigid body on a perfectly smooth horizontal plane. PMM Vol.45, No.3, 1981.
3. ZIGLIN S.L., Splitting of the separatrices, branching of the solutions and non-existence of an integral in the dynamics of a rigid body. Tr. Mosk. matem. o-va, Vol.41, 1980.
4. KOZLOV V.V. and ONISHCHENKO D.A., Non-integrability of Kirchhoff's equations. Dokl. Akad. Nauk SSSR, Vol.266, No.6, 1982.

Translated by L.K.

PMM U.S.S.R., Vol.49, No.3, pp.389-392, 1985 0021-8928/85 $10.00+0.00
Printed in Great Britain Pergamon Journals Ltd.

PERTURBED MOTION OF A KOVALEVSKAYA TOP*

N.N. MOTORINA

Perturbation theory based on the application of Lie series, is used to study a special case of the motion of a rigid body about a fixed point. The equations written in action-angle variables are used in Hamiltonian form. The solutions are obtained in the form of trigonometric series with constant coefficients.

It is assumed that the distribution of mass in the body is clsoe to the distribution in the Kovalevskaya case and the centre of gravity of the body is situated fairly near to the fixed point. The canonical Deprit variables /1/ are used. The motion of the body can be described in these variables by the following set of equations:

$$\frac{d(L, G, H)}{dt} = \frac{\partial F}{\partial(l, g, h)}, \quad \frac{d(l, g, h)}{dt} = -\frac{\partial F}{\partial(L, G, H)}$$

Using the condition that the centre of gravity of the body is situated fairly close to the fixed point and the principal moments of inertia A and B differ from each other, we can

**Prikl.Matem.Mekhan.*, 49, 3, 504-506, 1985